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Reduction of dimensions in random, elastic soil medium

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Abstract

The paper deals with an application of the plane strain analysis in a stochastic three-dimensional soil medium. In a framework of random elasticity theory, the geostatical state of stresses and the problem of a unit force acting in a statistically homogeneous half-space are considered. Only the modulus of elasticity is considered to be random and is modelled as a three-dimensional (3-D) homogeneous random field. As the result of imposed constrains due to the plane strain assumption the additional body and surface forces are induced. In order to determine them, additional equations must be introduced. The equations in a form of constrain relations are proposed in this paper. These equations are also valid for a case of uniformly distributed external loading.

First, the two-dimensional (2-D) problem and its reduction to the uni-axial strain state, for the gravity forces and uniform, unlimited surface loading is considered. Then, it is generalised into a 2-D schematization of the 3-D state. Next, the problem of a unit force acting in a statistically homogeneous half-space is considered. For a 3-D state of stress and strain the resulting stresses are compared with those for a 2-D state. These stresses for the multidimensional state of strain and stress are presented as a sum of two components. The first one reflects plane strain state stresses and is given in a form of a 3-D random field. This term allows for incorporating a spatial, 3-D soil variability into a two-dimensional analysis. The second component can be treated as a correction term and it represents the longitudinal influence of a 3-D analysis.

Some numerical results are presented in this paper. The proposed method can be regarded as a framework for further research aiming at application to a variety of geotechnical problems, for which the plane strain state is assumed.

1. Introduction

Recently, geotechnical problems involving uncertainties are receiving considerable attention from researchers and engineers. Due to the natural variability of most soils, limited number of observations, accuracy of measurements, etc., there exists a significant uncertainty with respect to soil parameters. Consequently, the results of such analysis as settlement, bearing capacity, or slope stability, are uncertain too. In order to develop a rational framework, which takes into consideration the uncertainty of material, the analysis must be reformulated into a stochastic one. It means that relationships between statistical characteristics of a random input and a random output must be determined.

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Solutions for various loads applied to a homogeneous (Green and Zerna, 1968; Leipholz, 1974) and non-homogeneous elastic half-space (Gibson, 1967; Kassir, 1972; Carrier and Christian, 1973; Lomakin, 1976), have played an important role in the development of foundation engineering. In a case of stochastic input the elasticity theory becomes random, so, the relations between input and output formally take a form of stochastic partial differential equations with random coefficients.

For an isotropic and homogeneous elastic soil medium the Young's modulus and the Poisson's ratio can be taken as the elastic parameters. The spatial variability of these parameters can be efficiently modelled as a multivariate and multidimensional random field (Wilde 1981; Vanmarcke, 1983).

A solution of the stochastic partial differential equations governing random elasticity leads to displacements and stresses, which are also multivariate, 3-D random fields. This solution, in most cases, is obtained in a numerical way, mainly using the stochastic finite element method. Many variants of this method have been developed recently (Bucher and Shinozuka, 1988; Deodatis, 1989; Liu et al., 1986; Shinozuka, 1987; Spanos and Ghanen, 1989; Yamazaki and Shinozuka, 1988). Also approximated analytical methods are available, like the perturbation procedure (imposing small fluctuation assumption) or Adomian's decomposition method (Eimer, 1972; Zeitoun and Baker, 1988). This group of methods has been a subject of some papers prepared by the author (Przewłócki, 1994, 1995).

In many geotechnical engineering problems, like retaining walls, strip foundations, or slopes and embankments (Fig. 1), the plane strain analysis is widely used. Such analysis is reasonable for elongated bodies of uniform cross-section subjected to a uniform loading along their longitudinal axes (x_1). It means that in the stochastic soil medium there is a full correlation in this direction, or in other words, a random variable model is assumed. Usually this is not so, and in some soils, depending on their origin, there can be even a significant horizontal variability of the material elastic parameters. Including such variabilities in the numerical analysis would require a 3-D stochastic finite element method. This method is not always available and its application is possible on a fast computer. Thus, the question of the validity of plane strain assumption for 3-D random fields of those parameters arises.

The main aim of the paper is the reduction of dimensions in the stochastic medium. The following basic questions should be answered:

(1) When is the plane strain schematization of the 3-D stochastic medium justified?
(2) It is possible to take into account 3-D spatial variability of soil properties in the plane strain analysis, and how to do it?

In order to answer them, an attempt is made to express stresses in the real state in terms of stresses in the reduced state. First, the 2-D problem and its reduction to the uni-axial state, for the gravity forces is considered. Then, this is generalised into 2-D schematization of the 3-D state. Next, the problem of a unit force acting in a statistically homogeneous half-space is considered. For a 3-D state of stress and strain the resulting stresses are compared with those for a 2-D state.

2. Stochastic soil medium

Different statistical models are proposed in the literature to describe a stochastic soil medium. Basically, they can be grouped into random variable models (Lumb, 1974) and random field

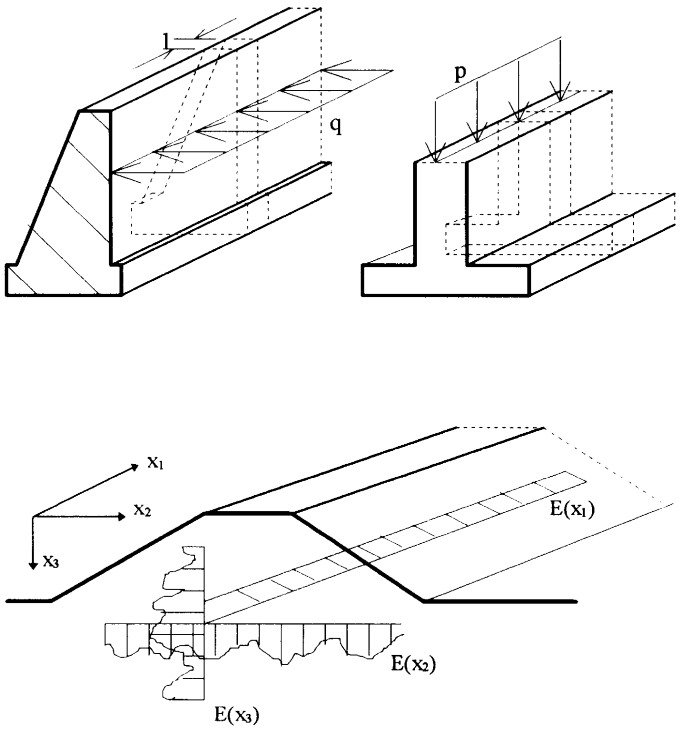

Fig. 1. Examples of elongated geotechnical structures.

models (Wilde, 1981; Vanmarcke, 1983). In elementary probability theory, the first models describe outcomes of experiments. In stress and deformation analysis performed in the framework of the elasticity theory, usually Young's modulus is taken as a random variable. In a second-order description, such a variable may be defined by its mean and standard deviation. Observing the outcome of all experiments is equivalent to observing the realisation of the random field. So the random field may be seen as the indexed family of random variables. The concept of random fields allows one to describe the spatial variability of any soil parameter. Again, in the second-order representation, the random field can be characterised by an average value, standard deviation, and an autocorrelation function, which describe the spatial dependence between the values of soil parameters at different points in space. The random variable model can be viewed as a limiting case of the random field theory, for the autocorrelation distance approaching infinity (full correlation).

It is assumed in this paper that the soil medium is a linearly elastic and isotropic body, so its response is defined by two elastic parameters: Young's modulus E and Poisson's ratio v. Randomness of these parameters influence the distribution and variance of stresses. It is well known that the relative variation of Poisson's ratio is much smaller than the variation of modulus of elasticity. To find the influence of both these parameters on variation of stress distribution a numerical calculation using stochastic finite element method were performed. A horizontal stratum of thickness $h = 20$ m subjected to uniformly distributed external loading of intensity $p = 10$ kPa was considered. The results, in dimensionless form, are presented in Fig. 2. They were performed

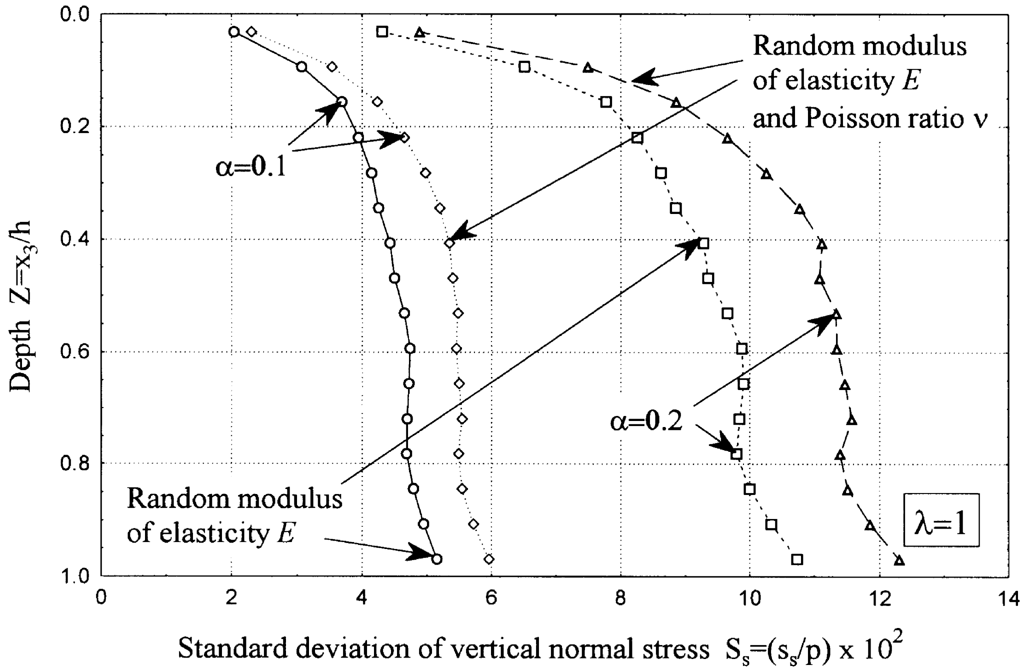

Fig. 2. Change of standard deviation of vertical normal stress with depth, regarding randomness only of E and both ν and E.

for covariance function given by (2) and (3), for following data: average values of elastic $\bar{E} = 100$ MPa, $\bar{\nu} = 0.3$, decay coefficient $\lambda = 1$ and for two values of coefficient of variation $\alpha = 0.1$ and $\alpha = 0.2$.

Separate calculations were performed: two for randomness of both parameters E and ν and two when only the modulus of elasticity was treated as a random field and the Poisson's ratio was assumed as a deterministic value. It is visible in Fig. 2 that the randomness of the Poisson's ratio has a secondary influence on the standard deviation of stress and for the real values of Poisson's ratio coefficients of variation, which are generally not higher than 0.1, it can be omitted.

It is further assumed that, for simplicity, ν is taken as a deterministic constant, while E is a homogeneous random field, which for a 3-D space, can be presented in the following form:

$$E = \bar{E} + \tilde{E}(x_1, x_2, x_3) = \bar{E}[1 + \alpha\beta(x_1, x_2, x_3)] \tag{1}$$

where α is coefficient of variation, $\beta(x_1, x_2, x_3)$ is normalized, homogeneous random field, $\langle\beta(x_1, x_2, x_3)\rangle = 0$, $\mathrm{Var}\,[\beta(x_1, x_2, x_3)] = 1$, $\langle\ldots\rangle$ is averaging operator.

The form of presentation of random field of the elasticity modulus given by eqn (1) and its application in numerical analysis is generally justified if the coefficient of variation α is sufficiently small, usually less than 0.15.

For the convenience of the further analysis, only the separable correlation structure is considered, so the random field is assumed to be statistically anisotropic. Such an assumption has been

Table 1
Results of correlation analysis

Correlation function	Coefficients		Loss function	
	$\tau-T$	$\tau-m$	$\tau-T$	$\tau-m$
$R(\rho)=e^{-\beta\cdot\rho}$	$\beta=5.023$	$\beta=1.886$	2.031	5.949
$R(x,y)=e^{-\beta_x\cdot\lvert x\rvert-\beta_y\cdot\lvert y\rvert}$	$\beta_x=4.209$	$\beta_x=1.944$	2.123	5.275
	$\beta_y=4.675$	$\beta_y=1.107$		
$R(x,y)=e^{-\beta_x\cdot\lvert x\cos\alpha+y\cdot sin\alpha\rvert-\beta_y\cdot\lvert x\cdot\cos\alpha+y\cdot\sin\alpha\rvert}$	$\beta_x=4.881$	$\beta_x=2.049$	1.982	4.947
	$\beta_y=2.908$	$\beta_y=0.971$		
	$\alpha=6.919$	$\alpha=0.176$		
$R(\rho)=(1+\beta\cdot\rho)\cdot e^{-\beta\cdot\rho}$	$\beta=10.061$	$\beta=3.588$	2.259	6.788
$R(x,y)=(1+\beta_x\cdot\lvert x\rvert)\cdot e^{-\beta_x\cdot\lvert x\rvert}\cdot(1+\beta_y\cdot\lvert y\rvert)\cdot e^{-\beta_y\cdot\lvert y\rvert}$	$\beta_x=9.407$	$\beta_x=4.049$	2.314	6.446
	$\beta_y=9.931$	$\beta_y=2.617$		
$R(x,y)=(1+\beta_x\cdot\lvert x\cdot\cos\alpha+y\cdot\sin\alpha\rvert)\cdot e^{-\beta_x\cdot\lvert x\cdot\cos\alpha+y\cdot\sin\alpha\rvert}$ $\times(1+\beta_y\cdot\lvert y\cdot\cos\alpha+x\cdot\sin\alpha\rvert)\cdot e^{-\beta_y\cdot\lvert y\cdot\cos\alpha+x\cdot\sin\alpha\rvert}$	$\beta_x=10.89$	$\beta_x=4.246$	2.275	6.249
	$\beta_y=7.655$	$\beta_y=2.416$		
	$\alpha=0.946$	$\alpha=0.226$		
$R(x,y)=e^{-\beta_x\cdot\lvert x\rvert-\beta_y\cdot\lvert y\rvert-\beta\cdot\lvert x\rvert\cdot\lvert y\rvert}$		$\beta_x=2.014$		5.258
		$\beta_y=1.126$		
		$\beta=-0.170$		

where $\rho=\sqrt{x^2+y^2}$, $x=x_i-x_j$, $y=y_i-y_j$.

confirmed by extensive in situ tests performed by the author. Several correlation functions were used for approximation of obtained results. They are presented in Table 1.

The global and the local approaches to the description of soil randomness were considered. In the global approach, deviations are calculated subtracting mean value (m) for the measured values (τ) and in the local approach a trend (T) is subtracted from the realisation (τ). Loss function is a sum of squared deviations. Its come from the performed analysis that several correlation functions could be used to approximate the test results. One of them is a function of a separable correlation structure which is especially convenient in further analysis.

Due to equations describing the random elasticity, the random field of the modulus of elasticity must be differentiable, so the following covariance function can be taken from Table 1 for further considerations:

$$R_\beta(\tau_1,\tau_1,\tau_3) = R_1(\tau_1)R_2(\tau_2)R_3(\tau_3) \tag{2}$$

where

$$R_i(\tau_i) = (1+\lambda_i\tau_i)\,e^{-\lambda_1\tau_i}, \quad i=1,2,3, \quad \tau_i \geqslant 0 \tag{3}$$

$\tau_i = \lvert x_i'-x_i''\rvert$, λ_i—correlation decay coefficient.

The correlation distance ($b_i = 1/\lambda_i$), instead of the correlation decay coefficient is commonly used in the literature.

Presented in Table 1 are the results obtained for one type of soil viz silty clay, so they cannot form a proof that the separable correlation structure is valid for all types of soil. Unfortunately the author could not find in the literature any information referring to this problem. So at the present stage, the validity of the separate correlation structure for other soils should be regarded only as an assumption.

3. Random elasticity equations

The states of stress and strain at each point of the elastic soil medium are fully described by stress σ_{ij} and strain ε_{ij} tensors. For small deformations, the strain tensor is defined in the following way:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2,3 \tag{4}$$

where u_i is a displacement vector.

Here the Einstein summation notation is applied.

The stress tensor should satisfy the following equilibrium conditions:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = -X_i \tag{5}$$

where X_i represent body forces.

Stresses and strain are related to each other through the material law. For the elastic continuum, the Hooke's law is valid, which for multidimensional state of stress and strain can be written in the form:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad i,j,k,l = 1,2,3 \tag{6}$$

where C_{ijkl} is an elasticity tensor.

For a homogeneous and isotropic medium, the elasticity tensor, in terms of Lamé's constants λ and μ, can be written as follows:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jl}\delta_{jk}) \tag{7}$$

where δ_{ij} is the Kronecker's symbol.

In the case of a stochastic soil medium, treated as a random elastic continuum, the elastic parameters are not constant anymore. They vary from point to point, so they are functions of locations. Thus, we can write:

$$C_{ijkl} = C_{ijkl}(x_1, x_2, x_3), \quad \lambda = \lambda(x_1, x_2, x_3), \quad \mu = \mu(x_1, x_2, x_3) \tag{8}$$

Substituting (4), (6) and (7) into (2), and taking into account the elasticity tensor treated as the random field (8), the second-order stochastic differential equations in displacements are obtained. Eventually they can be written in a following form:

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_i}+\mu\nabla^2 u_i+\theta\frac{\partial\lambda}{\partial x_i}+\frac{\partial\mu}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=X_i \tag{9}$$

where

$$\theta=\frac{\partial u_k}{\partial x_k},\quad \nabla^2=\frac{\partial^2}{\partial x_i\,\partial x_i}$$

In order to solve eqns (9), the boundary conditions for a given problem must be introduced. They can be presented either for specified surface forces p_i or specified surface displacements u_{i0} as follows:

$$\sigma_{ij}n_j=p_i,\quad u_i(0)=u_{i0} \tag{10}$$

where n_j are components of a vector normal to the surface.

As a result the vectorial random field of displacements is obtained and it can be described by average values and a correlation function:

$$u_k(x_1,x_2,x_2),\quad \bar{u}_k(x_1,x_2,x_2),\quad R_{u_k}(x_1',x_2',x_3',x_1'',x_2'',x_3'') \tag{11}$$

Finally, the tensorial random field of stresses can be found:

$$\sigma_{ij}(x_1,x_2,x_2),\quad \bar{\sigma}_{ij}(x_1,x_2,x_2),\quad \mathrm{Var}\,[\sigma_{ij}(x_1,x_2,x_2)] \tag{12}$$

In the present paper, the description of this field is limited only to its mean value and the variance.

It is worth noting that the eqns (9) are stochastically non-linear, because there is a multiplication of two random fields (Lamé's parameters and displacement vector).

4. Plane strain analysis in a stochastic medium

In a case of the plane strain state, the displacement in one direction (longitudinal) is constant or in a limiting case it vanishes, while other displacements are functions of only two variables:

$$u_1=u_1(x_1,x_2),\quad u_2=u_2(x_1,x_2),\quad u_3=c \tag{13}$$

According to eqn (4) there are the following, also 2-D, non-zero strains:

$$\varepsilon_{11}=\varepsilon_{11}(x_1,x_2),\quad \varepsilon_{22}=\varepsilon_{22}(x_1,x_2),\quad \varepsilon_{12}=\varepsilon_{12}(x_1,x_2) \tag{14}$$

The strains are related to stresses through the elastic parameters (6), which according to (8) are 3-D. Thus, the stresses in the plane strain state are also 3-D, and can be presented in the following form:

$$\begin{aligned}&\sigma_{11}^{\mathrm{II}}=\sigma_{11}^{\mathrm{II}}(x_1,x_2,x_3),\quad \sigma_{22}^{\mathrm{II}}=\sigma_{22}^{\mathrm{II}}(x_1,x_2,x_3)\\&\sigma_{12}^{\mathrm{II}}=\sigma_{12}^{\mathrm{II}}(x_1,x_2,x_3),\quad \sigma_{33}^{\mathrm{II}}=\sigma_{33}^{\mathrm{II}}(x_1,x_2,x_3)\end{aligned} \tag{15}$$

The remaining stresses are equal to zero.

The presentation of stresses are given by (15) allows one to introduce, in a formal way, 3-D soil medium variability into the plane strain analysis.

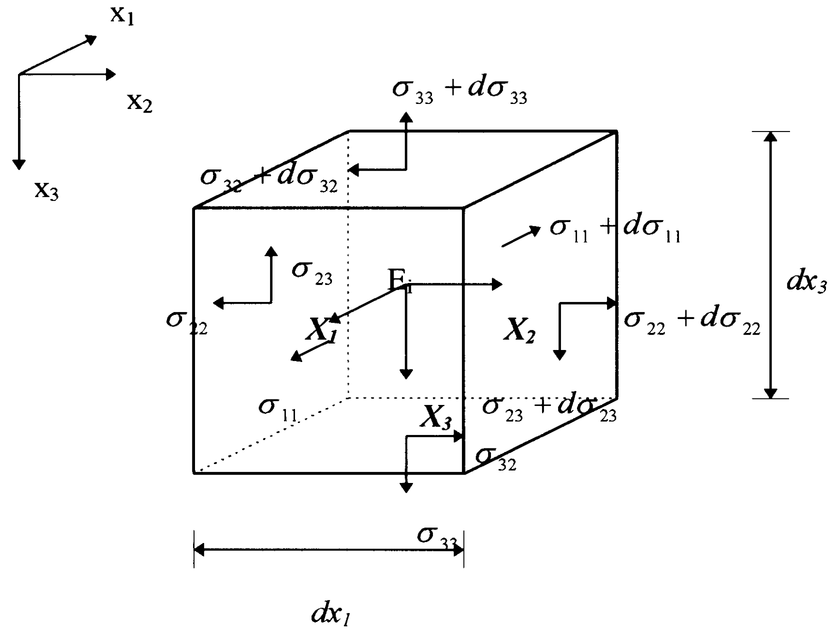

Fig. 3. Elementary stresses in the plane strain state.

Considering the equilibrium eqns (5), it is seen that in the 2-D strain state the third equation is not identically equal to zero. It implies the existence of a non-zero body force in this direction. The stresses and body forces acting at the elementary cubicoid of unit width are shown in Fig 3. In general, the plane strain state can be modelled in the arbitrary non-homogeneous medium provided that additional body X_i and surface forces p_i are applied.

These forces are caused by constrains due to the plane strain assumption and do not allow for the longitudinal displacements. They are unknown and result from the equilibrium equations and boundary conditions.

$$X_1 = -\frac{\partial}{\partial x_1}\left[\lambda\left(\frac{\partial u_2}{x_2}+\frac{\partial u_3}{\partial x_3}\right)\right]$$

$$X_2 = -\frac{\partial}{\partial x_2}\left[\lambda\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+2\mu\frac{\partial u_2}{\partial x_2}\right]-\frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\right]$$

$$X_3 = -\frac{\partial}{\partial x_3}\left[\lambda\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+2\mu\frac{\partial u_3}{\partial x_3}\right]-\frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\right] \quad (16a)$$

$$p_2 = \lambda\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)$$

$$p_3 = \lambda\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+2\mu\frac{\partial u_3}{\partial x_3} \quad (16b)$$

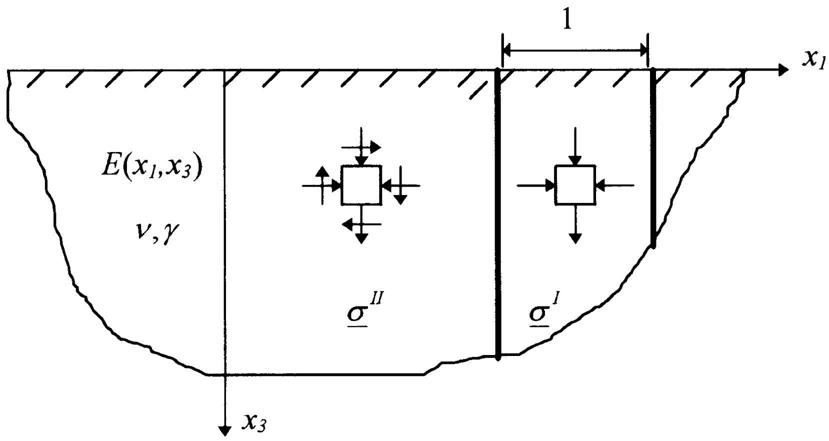

Fig. 4. Half-space in the plane strain and the uni-axial strain states.

It should be emphasised that in interpretation of the plane strain state in a stochastic medium cannot be identified with a case of soil between two rigid plates close to each other. Such constrains are imposed on every point within the soil medium.

Both body and surface forces are expressed in terms of displacements. In fact, eqns (16a) and (16b) are not independent, so there are eventually $n = 11$ unknowns and $r = 9$ equations. In the homogeneous medium there are eight unknowns and eight equations. So, two additional equations must be introduced.

5. Uni-axial strain state

For simplicity and better understanding, the 2-D problem and its reduction to the uni-axial strain state, for the gravity forces only, is considered. The stresses acting at the elementary rectangles in both states are shown in Fig. 4. Here, the Young's modulus is assumed to be a 2-D random field.

In the uni-axial strain state, the displacement and the strain depend only on x_3:

$$u_3 = u_3(x_3), \quad u_2 = u_1 = 0$$
$$\varepsilon_3 = \varepsilon_{33}(x_3), \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{11} = \varepsilon_{22} = 0 \tag{17}$$

but stresses are functions of two variables:

$$\sigma^{\mathrm{I}}_{33}(x_1, x_3) = AE(x_1, x_3)\varepsilon_{33}(x_3), \quad \sigma^{\mathrm{I}}_{11} = \frac{v}{1-v}\sigma_{33} \tag{18}$$

where

$$A = \frac{(1+\nu)(1-2\nu)}{(1-\nu)}$$

There are two equilibrium equations in the uni-axial strain state:

$$X_i = -\frac{\partial \sigma^{\mathrm{I}}_{11}}{\partial x_1} = -A\frac{\partial}{\partial x_1}[E(x_1,x_3)\varepsilon_{33}(x_3)]$$

$$X_3 = -\frac{\partial \sigma^{\mathrm{I}}_{33}}{\partial x_3} = -A\frac{1-\nu}{\nu}\frac{\partial}{\partial x_3}[E(x_1,x_3)\varepsilon_{33}(x_3)] \tag{19}$$

These equations are mutually dependent, so eventually, there are $n = 4$ unknown variables u_3, ε_{33}, σ_{33}, X_3 and $r = 3$ independent equations (static, geometric and constitutive). Thus, there must be introduced one additional equation.

Let us write the Hook's law (6) in the following form:

$$\bar{\sigma}[1+\alpha f(x_1,x_3)] = A\bar{E}[1+\alpha\beta(x_1,x_3)]\bar{\varepsilon}[1+\alpha\beta^*(x_3)] \tag{20}$$

where each component is presented as a sum of the average (dashed) and the fluctuated parts.

The functions $f(x,x_3)$ and $\beta^*(x_3)$ are unknown, whereas the function $\beta(x_1,x_3)$ represents the input random field of the Young's modulus. Equating the terms with the same power of α gives two equations:

$$\bar{\sigma} = A\bar{E}\bar{\varepsilon} = \gamma x_3$$

$$f(x_1,x_3) = \beta(x_1,x_3)+\beta^*(x_3) \tag{21}$$

The first one corresponds to the average solution and it is a well-known expression determining vertical stresses in the geostatical state.

In order to solve the second equation, two limiting cases are considered (Fig. 5):

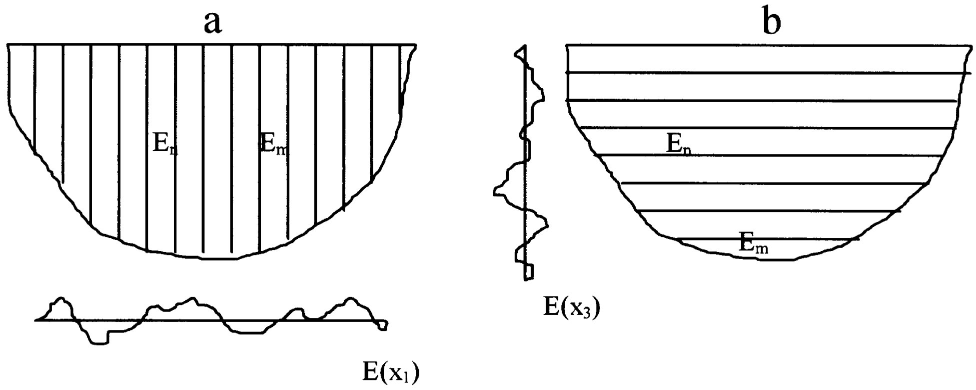

Fig. 5. Limiting cases of soil spatial variability (a) vertical stratification, (b) horizontal stratification.

5.1. Full correlation in a vertical direction

This case can be thought of as a medium consisting of thin vertical strata with identical densities but different moduli of elasticity, which vary only with x_1, so $\beta(x_1, x_3) = \beta_1(x_1)$. In the uni-axial strain state there is no strain fluctuation $\beta^*(x_3) = 0$ and so the unknown function is $f(x_1, x_3) = \beta_1(x_1)$. The stress in this state can be written as follows:

$$\sigma_{33}^{\mathrm{I}}(x_1, x_3) = \gamma x_3[1 + \alpha\beta_1(x_1)] \tag{22}$$

5.2. Full correlation in a horizontal direction

In this case the medium is built up of the horizontal thin layers and the Young's modulus varies only with x_3, so $\beta(x_1, x_3) = \beta_3(x_3)$. It is obvious here that the unknown function must be equal to zero $f(x_1, x_3) = 0$ (strains are deterministic), so there is a full but negative correlation between fluctuations of strain and the elasticity modulus. The vertical stress is just a geostatistical one:

$$\sigma_{33}^{\mathrm{I}}(x_1, x_3) = \bar{\sigma} = \gamma x_3 \tag{23}$$

5.3. General case

It is seen that there are two different expressions for stresses (22) and (23), depending on the kind of non-homogeneity represented by parameters describing elasticity. Thus, an equation describing the non-homogeneity must be introduced. For full correlation in vertical direction $\beta^*(x_3) = 0$, while for full correlation in horizontal direction $\beta^*(x_3) = -\beta(x_3)$. An attempt was made to combine the above in one expression in possible simple form. The best way of doing it is to assume that the relation between $\beta^*(x_3)$ and $\beta(x_3)$ can be expressed in the following way:

$$\beta^*(x_3) = -D(\lambda_1, \lambda_3)\beta_3(x_3) \tag{24}$$

In fact, this is a constitutive type equation, because it relates one of the basic values, in this case strain's fluctuations, with the parameters describing non-homogeneity of the material. The eqn (24) can be also thought of as the constrain relation, resulting from the imposed model constrain due to plane strain assumption. The function D, for the considered limiting cases, varies from 0–1, and it depends on the correlation decay coefficients λ_1, λ_3. The following, possible simple relationship is proposed:

$$D(\lambda_1, \lambda_3) = \frac{\lambda_3}{(\lambda_1 + \lambda_3)^2} \times \left[\lambda_3 \cdot \exp\left(-a \cdot \frac{\lambda_1^2}{\lambda_3}\right) + 3\lambda_1 \cdot \exp(-a \cdot \lambda_1)\right] \tag{25}$$

where a is some undefined parameter.

It is worth noting that the formulae (25) is also valid for other limiting cases i.e. for a lack of correlation in either horizontal or vertical direction. For example, the lack of correlation in the vertical direction can be understood as a medium consisting of strata of infinite small thickness. Thus, it is easy to prove that $D \to 1$ for $\lambda_3 \to \infty$. On the other hand, for the lack of correlation in the horizontal direction, we have $D \to 0$ for $\lambda_1 \to \infty$. In the special case when $\lambda_1 = \lambda_3 = \lambda$ we obtain $D(\lambda_1, \lambda_3) = \exp(-a \cdot \lambda)$. For the case of random field model i.e. $\lambda_1 = \lambda_3 = 0$ and $a \neq 0$ we have

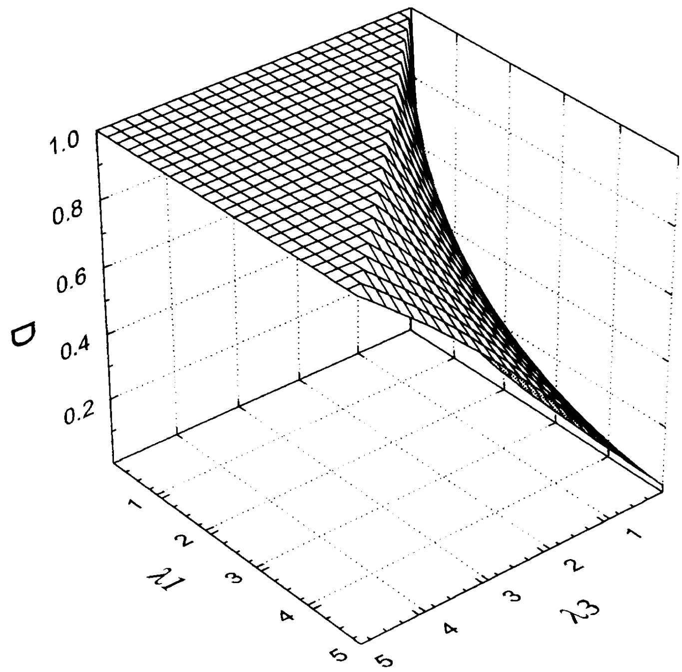

Fig. 6. Graphical interpretation of the function $D(\lambda_1, \lambda_3)$, for parameter $a = 0$.

$D = 1$, what was expected. Equation (25) does not include spatial variability of elasticity modulus if parameter $a = 0$.

The relationship (25) can be presented graphically as a 3-D plot. For a parameter $a = 0$ it is shown in Fig. 6, whereas for $a = 1$ in Fig. 7.

Taking into account (24), the expression for the stress (20) can be presented in the following form:

$$\sigma^{\mathrm{I}}_{33}(x_1, x_3) = \bar{\sigma}^{\mathrm{I}}_{33}\{1 + \alpha\beta_3(x_3)[\beta_1(x_1) - D]\} \tag{26}$$

The above presentation of the stress in the uni-axial strain state allows for an explicit incorporation of 2-D random field of the Young's modulus into the 1-D analysis. The average stress is equal to the geostatical one, and its variance can be easily found. Finally, it can be written in the form:

$$\mathrm{Var}\,[\sigma^{\mathrm{I}}_{33}(x_1, x_3)] = (\alpha\gamma x_3)^2(1 - D)^2 \tag{27}$$

For the case of constrain relationship given by (25), the variance of vertical stress, in the uni-axial strain state, is given by the following expression:

$$\mathrm{Var}\,[\sigma^{\mathrm{I}}_{33}(x_1, x_3)] = (\alpha\gamma x_3)^2 \cdot \left\{1 - \frac{\lambda_3}{(\lambda_1 + \lambda_3)^2} \times \left[\lambda_3 \cdot \exp\left(-a \cdot \frac{\lambda_1^2}{\lambda_3}\right) + 3\lambda_1 \cdot \exp(-a \cdot \lambda_1)\right]\right\}^2 \tag{28}$$

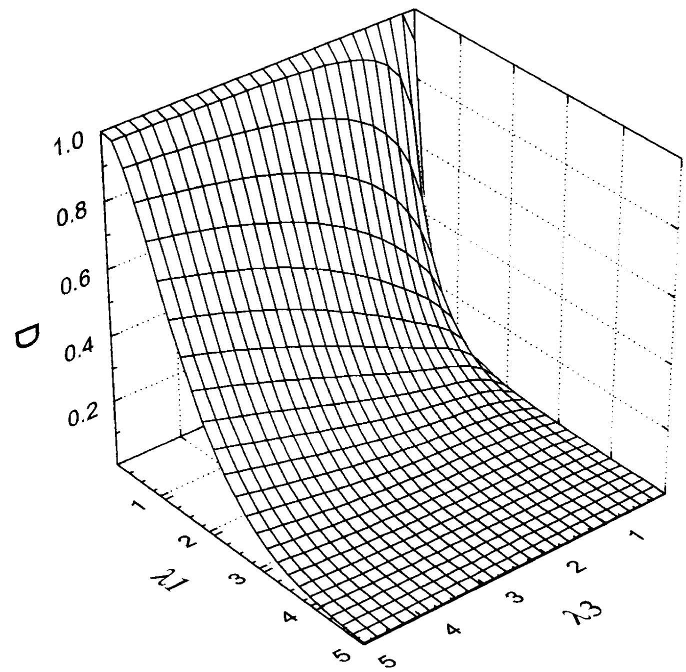

Fig. 7. Graphical interpretation of the function $D(\lambda_1, \lambda_3)$, for parameter $a = 1$.

It is worth noting that the variance approaches zero in three cases:

$$\mathrm{Var}\,[\sigma^{\mathrm{I}}_{33}] \to 0: \quad \alpha \to 0, \quad \lambda_1 \to 0, \quad \lambda_3 \to \infty \tag{29}$$

i.e. if the variability of the Young's modulus is small, if there is a full correlation in the horizontal direction or lack of correlation in the vertical direction. In fact, these cases define a deterministic homogeneity of the soil medium or its horizontal stratification. For such cases, stresses in the uniaxial strain state are the same as the corresponding stresses in the plane strain analysis.

Thus, the variance of the stress (27) can be thought of as a criterion of validity of the dimensions's reduction. At the present stage of study on that problem, one can say that the reduction is justified if the variance is small enough. How small, it is a matter of some quantitative studies.

In order to determine the parameter a, numerical calculations based on the stochastic finite element method were performed. The horizontal soil stratum subjected to gravity and only most important in the practice case i.e. $\lambda = \lambda_1 = \lambda_3$ was considered. The input data are given in Chapter 2 of the present paper. The results, i.e. the variance of the vertical normal stress vs decay correlation coefficient are presented in Fig. 8. The computations were performed for four different values of decay coefficient $\lambda = 1, 2, 5$ and 10, each for several depths. They allowed to estimate the parameter $a = 0.7$ and then to find the variance of the vertical normal stress according to (28), which relation is also shown in Fig. 8.

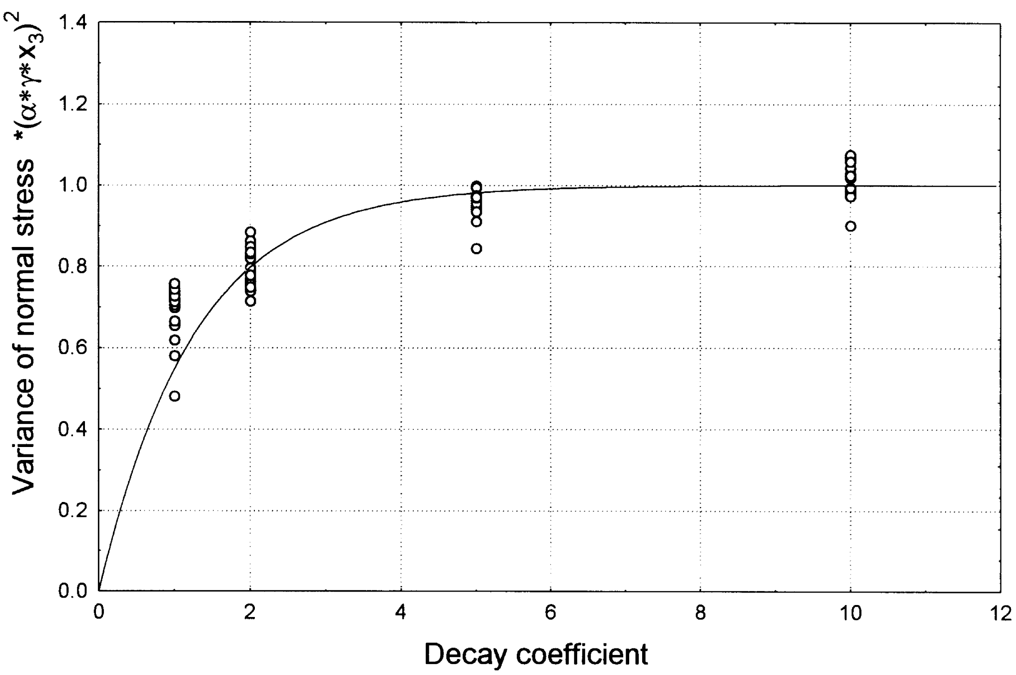

Fig. 8. Variance of vertical normal stress vs correlation decay coefficient.

6. The generalisation into the plane strain analysis

The previous considerations can be generalised into 2-D reduction of a 3-D problem.

For such reduction, two additional equations must be introduced. They can be obtained by considering three limiting cases shown in Fig. 9.

In a similar way as before, the following constitutive type relationships are proposed:

$$\varepsilon_{22} = \bar{\varepsilon}_{22}[1-D_1\beta(x_2,x_3)]$$
$$\varepsilon_{33} = \bar{\varepsilon}_{33}[1-D_2\beta(x_2,x_3)] \tag{30}$$

Equations (30) is contrary to eqn (24) relating explicitly strains with the material's parameters. Now, using (25), there are two parameters given in the form:

$$D_1(\lambda_1,\lambda_2) = \frac{\lambda_2}{(\lambda_1+\lambda_2)^2}\times\left[\lambda_2\cdot\exp\left(-a\cdot\frac{\lambda_1^2}{\lambda_2}\right)+3\lambda_1\cdot\exp(-a\cdot\lambda_1)\right]$$
$$D_2(\lambda_1,\lambda_3) = \frac{\lambda_3}{(\lambda_1+\lambda_3)^2}\times\left[\lambda_3\cdot\exp\left(-a\cdot\frac{\lambda_1^2}{\lambda_3}\right)+3\lambda_1\cdot\exp(-a\cdot\lambda_1)\right] \tag{31}$$

Equations (30) together with basic eqns (4)–(6) form a set of $n = 11$ equations with 11 unknown variables. Its solution is not in the scope of this paper.

It is worth noting that a similar set of 11 equations may be obtained in a framework of the

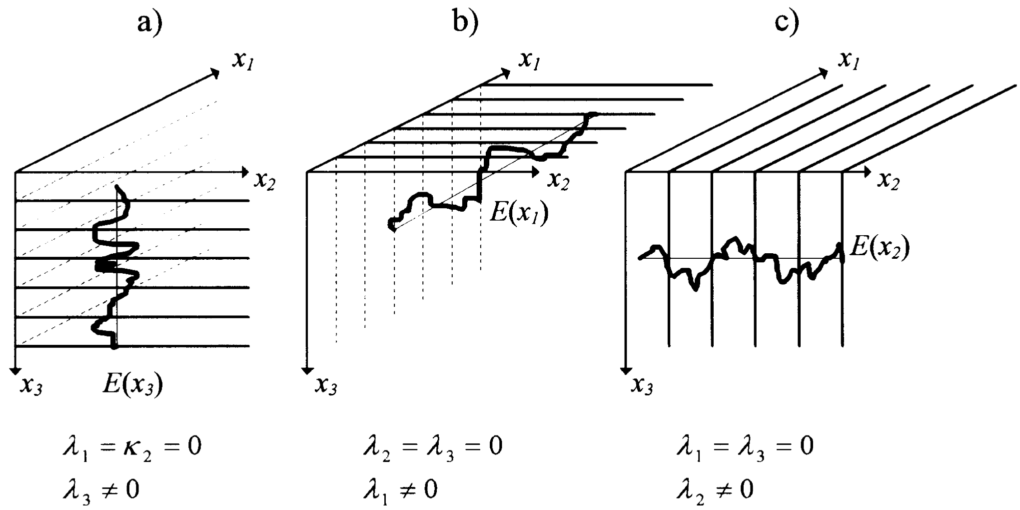

Fig. 9. Limiting cases in the spatial 3-D soil medium.

theory of constrains. However, such additional equations, defining constrains, would be much more complicated than (30), and the solution of a general set of equations would be more difficult.

7. Stresses in 2-D and 3-D states

It has been shown that the stress tensor in plane strain analysis is the 3-D tensorial random field. It seems to be profitable to compare this field with the one from a solution of the spatial, 3-D problem. Of course, such comparison should be performed for some equivalent problems.

In the paper, the boundary value problems of a unit force acting in the random, elastic half-plane and half-space are considered. The stresses both in the plane strain state as well as in the 3-D state can be presented, using Green's functions, as sums of stochastic integrals. For instance, the normal stresses in the longitudinal (x_1 in Fig. 10) direction in those two states can be written as follows:

$$\tilde{\sigma}_1^{\mathrm{II}}(x_s) = \alpha A\bar{E}v \int\int_A \left\{ \frac{\partial}{\partial x_2}[G_{2_j}^{\mathrm{II}}(x_s,\xi_s) f_j^{\mathrm{II}}(\xi_s)] + \frac{\partial}{\partial x_3}[g_{3_j}^{\mathrm{II}}(x_s,\xi_s) f_j^{\mathrm{II}}(\xi_s)]\, \mathrm{d}A(\xi_s) \right\} \tag{32a}$$

$$\tilde{\sigma}_1^{\mathrm{III}}(x_s) = \alpha FE \iiint_V \left\{ (1-v)\frac{\partial}{\partial x_1}[G_{1_j}^{\mathrm{III}}(x_s,\xi_s) f_j^{\mathrm{III}}(\xi_s)] + v\frac{\partial}{\partial x_2}[G_{2_j}^{\mathrm{III}}(x_s,\xi_s) f_j^{\mathrm{III}}(\xi_s)] + v\frac{\partial}{\partial x_3}[G_{3_j}^{\mathrm{III}}(x_s,\xi_s) f_j^{\mathrm{III}}(\xi_s)] \right\} \tag{32b}$$

where

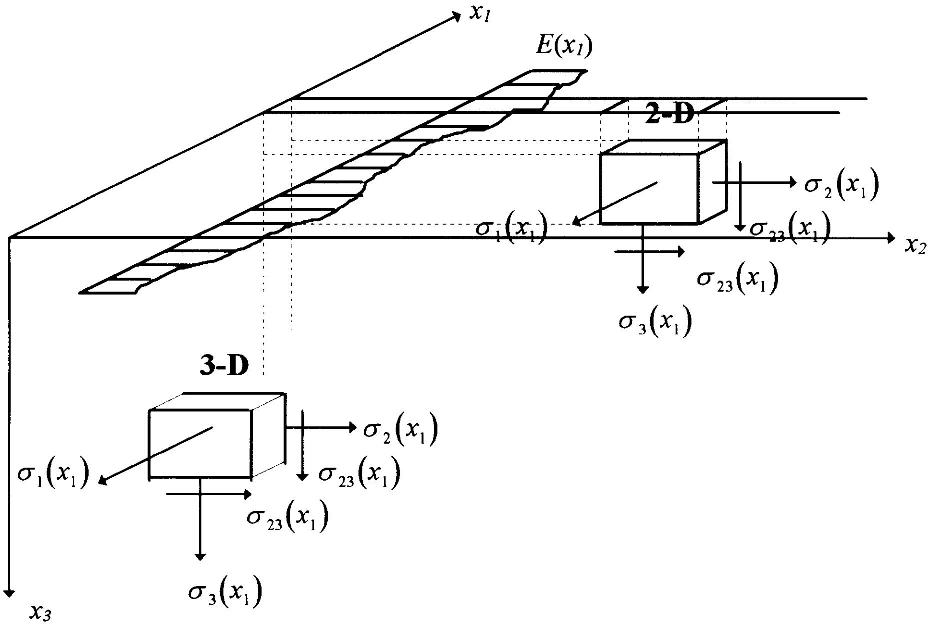

Fig. 10. Stresses in 2-D and 3-D states.

$$f_j = -\left[\beta X_j - \bar{\lambda}\theta_0 \frac{\partial \beta}{\partial x_j} - \bar{\mu}\frac{\partial \beta}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\mathrm{d}x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right)\right]$$

$$F = \frac{1}{(1+v)(1-2v)} \tag{33}$$

For the considered problem, Green's functions G_{ij} are known as the Melan's and Mindlin's solutions, respectively. Functions f_j are random and they involve the input random field of the Young's modulus. Depending on the superscript (II or III), they are 2-D ($j = 2, 3$) or 3-D fields.

The stresses in both states (in 3-D only with identical subscripts) are shown in Fig. 7. The components of 2-D and 3-D stresses having identical subscripts can be related to each other.

Let us compare the equivalent terms of (32). For instance in a case $i = k = 3$ and $j = l = 3$ they may be written as follows:

$$K = \frac{\partial}{\partial x_3}\iiint_V G_{33}^{\mathrm{III}}(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) f_3^{\mathrm{III}}(\xi_1, \xi_2, \xi_3)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$$

$$\Updownarrow$$

$$H = \frac{\partial}{\partial x_3}\iint_A G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3) f_3^{\mathrm{II}}(\xi_2, \xi_3)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \tag{34}$$

It is easy to prove the following identity:

$$J = \int_{-\infty}^{\infty} \frac{\partial}{\partial x_3} G_{33}^{II}(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)\,\mathrm{d}\xi = \frac{\partial}{\partial x_3} G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3)$$

$$= \frac{\partial}{\partial x_3} G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3) \frac{\int_{-\infty}^{\infty} \frac{\partial}{\partial k_3} G_{33}^{\mathrm{III}}(x_1, x_2, x_3, \xi_1, 0, 0\text{—}\,\mathrm{d}\xi}{\frac{\partial}{\partial x_3} G_{33}^{\mathrm{II}}(x_2, x_3, 0, 0)}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial x_3} \left[\frac{\frac{\partial}{\partial x_3} G_{33}^{\mathrm{III}}(x_1, x_2, x_3, \xi_1, 0, 0)}{\frac{\partial}{\partial x_3} G_{33}^{\mathrm{II}}(x_2, x_3, 0, 0)} G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3) \right] \mathrm{d}\xi_1$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial x_3} [g(x_1, x_2, x_3, \xi_1) G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3)]\,\mathrm{d}\xi_1 \tag{35}$$

A function $g(x_1, x_2, x_3, \xi_1)$ introduced in (35) is given by the formula:

$$g(x_1, x_2 x, {}_3, \xi_1) = \frac{\frac{\partial}{\partial x_3} G_{33}^{\mathrm{III}}(x_1, x_2, x_3, \xi_1, 0, 0)}{\frac{\partial}{\partial x_3} G_{33}^{\mathrm{II}}(x_2, x_3, 0, 0)} \tag{36}$$

It is assumed that this function is determined for constant values of ξ_2 and ξ_3 (for simplicity equal to zero), and must fulfil the following condition:

$$\int_{-\infty}^{\infty} g(x_1, x_2, x_3, \xi_1)\,\mathrm{d}\xi_1 = 1 \tag{37}$$

The assumption of statistical anisotropy:

$$\beta(\xi_1, \xi_2, \xi_3) = \beta_1(\xi_1)\beta_2(\xi_2, \xi_3) \tag{38}$$

allows one to present the random function appearing in (32) as follows:

$$f_j^{\mathrm{III}}(\xi_1, \xi_2, \xi_3) = \beta_1(\xi_1) f_j^{\mathrm{II}}(\xi_2, \xi_3) \tag{39}$$

Substituting (36) and (39) into (36), the term K can be presented in the following form:

$$K = \iiint_V \frac{\partial}{\partial x_3} [g(x_1, x_2, x_3, \xi_1) G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3)] \beta_1(\xi_1) f_3^{\mathrm{II}}(\xi_2, \xi_3)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$$

$$= \iiint_V \Big[G_{33}^{\mathrm{II}}(x_2, x_3, \xi_2, \xi_3) \frac{\partial}{\partial x_3} g(x_1, x_2, x_3, \xi_1)$$

$$+g(x_1,x_2,x_3,\xi_1)\frac{\partial}{\partial x_3}G_{33}^{\mathrm{II}}(x_2,x_3,\xi_2,\xi_3)\Bigg]\beta_1(\xi_1)f_3^{\mathrm{II}}(\xi_2,\xi_3)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$$

$$=\iint_A G_{33}^{\mathrm{II}}(x_2,x_3,\xi_2,\xi_3)f_3^{\mathrm{II}}(\xi_2,\xi_3)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\cdot\int_{\xi_1}\frac{\partial}{\partial x_3}g(x_1,x_2,x_3,\xi_1)\beta_1(\xi_1)\,\mathrm{d}\xi_1$$

$$+\iint_A\frac{\partial}{\partial x_3}G_{33}^{\mathrm{II}}(x_2,x_3,\xi_2,\xi_3)f_3^{\mathrm{II}}(\xi_2,\xi_3)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\cdot\int_{\xi_1}g(x_1,x_2,x_3,\xi_1)\beta_1(\xi_1)\,\mathrm{d}\xi_1 \quad (40)$$

Substituting function H given by the second expression of (36) into (40), the term K may be eventually determined by the following expression:

$$K=\left[\int H(x_2,x_3)\,\mathrm{d}x_3\right]\times\int_{\xi_1}\frac{\partial g}{\partial x_3}\beta_1(\xi_1)\,\mathrm{d}\xi_1+H(x_2,x_3)\int_{\xi_1}g\beta_1(\xi_1)\,\mathrm{d}\xi_1 \quad (41)$$

The term K is a component of stress in a 3-D analysis whereas the corresponding term H is a component of the stress in the plane strain analysis. The expression (41) explicitly relates these two terms.

Finally, the spatial stress can be presented as a sum of two components:

$$\tilde{\sigma}_{11}^{\mathrm{III}}(x_1,x_2,x_3)=\tilde{\sigma}_{11}^{\mathrm{II}}(x_1,x_2,x_3)+R(x_1,x_2,x_3) \quad (42)$$

The first one reflect the plane strain stress and may be given in the form of a 3-D random field. The second component can be treated as a correction term and it represents the longitudinal influence of the spatial analysis. It is given in the form of a triple stochastic integral.

The case of the generalisation into a plane strain analysis is rather difficult. However, the approach presented in the paper can be regarded as a framework for further theoretical and numerical research in a field of dimension's reduction in stochastic soil mechanics.

8. Case study

An attempt is presented to find a transfer function k between variances of stresses for 2-D and 3-D states. Such function permits to find the variances for 3-D state if a variance in 2-D state is known. It can be written as follows:

$$\mathrm{Var}\,[\tilde{\sigma}_{i,j}^{\mathrm{III}}(x_2,x_3)]=k\cdot\mathrm{Var}\,[\tilde{\sigma}_{ij}^{\mathrm{II}}(x_2,x_3)] \quad (43)$$

The above was performed for a strip foundation of width $B = 2.5$ m and intensity of loading $p = 10$ kPa, placed on a soil stratum of thickness $h = 20$ m, underlaid by a smooth, rigid base. The numerical calculations were performed both in plane strain state and in 3-D state. The following parameters characterising soil medium and its randomness were assumed: Poisson's ratio $v = 0.3$, expected value of elasticity modulus $\bar{E} = 100$ MPa, its coefficient of variation $\alpha_E = 0.1$ and three values of decay coefficient $\lambda = 1,2$ and 5. It was also assumed that the covariance function is given by (2) and (3). The comparison of solutions for both states allows to find out a suitable transfer function k that appears in formulae (43).

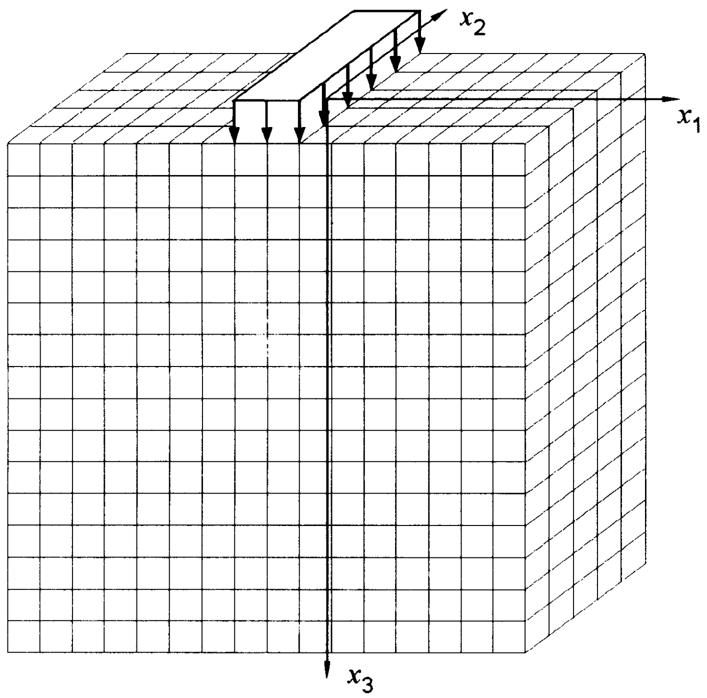

Fig. 11. Strip foundation and finite element mesh applied in 3-D analysis.

The computations were performed using modified version of program NONSAP for stochastic finite element method (FEM), where simulation procedure described by Bielewicz et al. (1996) was applied. The foundation and assumed 3-D mesh is presented in Fig. 11.

The obtained standard deviations of normal vertical stresses S_s^{II} and S_s^{III}, for decay coefficients $\lambda = 1$, $\lambda = 2$, $\lambda = 5$ [m^{-1}] were approximated by linear relationships. The relationship between standard deviations for 2-D and 3-D states, and for one decay coefficient $\lambda = 5$ is shown in Fig. 12. It can be written in a following form:

$$S_s^{\mathrm{III}} = b \cdot S_s^{\mathrm{II}} \tag{44}$$

where $b = 1.263$.

The results obtained for all considered decay coefficients are presented in Fig. 13. It is seen that the standard deviation of normal vertical stress for 3-D state varies exponentially. A good fitness was obtained for the function given by the following expression:

$$k = [\exp(c \cdot \lambda^n)]^2 \tag{45}$$

where the coefficients appearing in (45) were found to be equal $c = 0.16$, $n = 0.18$. The square root expressed by (45) transfer function is presented in Fig. 13.

9. Remarks and conclusions

The problem of reducing dimensions in the stochastic soil medium is analysed. In a framework of random elasticity the problems of the geostatistical state of stresses, uniform vertical surface

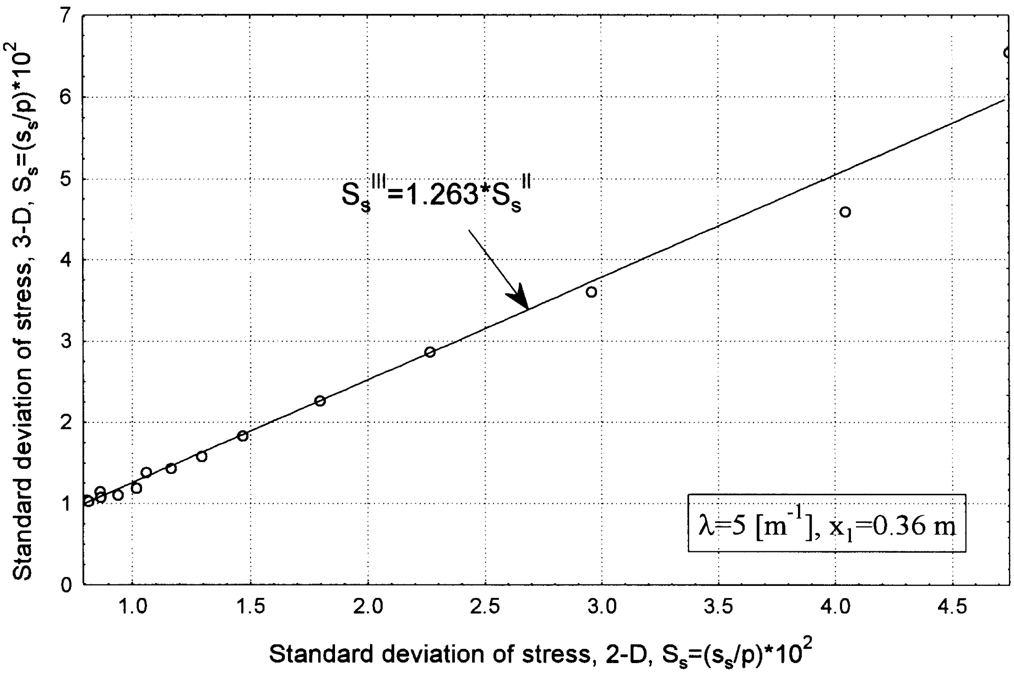

Fig. 12. Relationship between standard deviations of normal vertical stresses for 2-D and 3-D analyses.

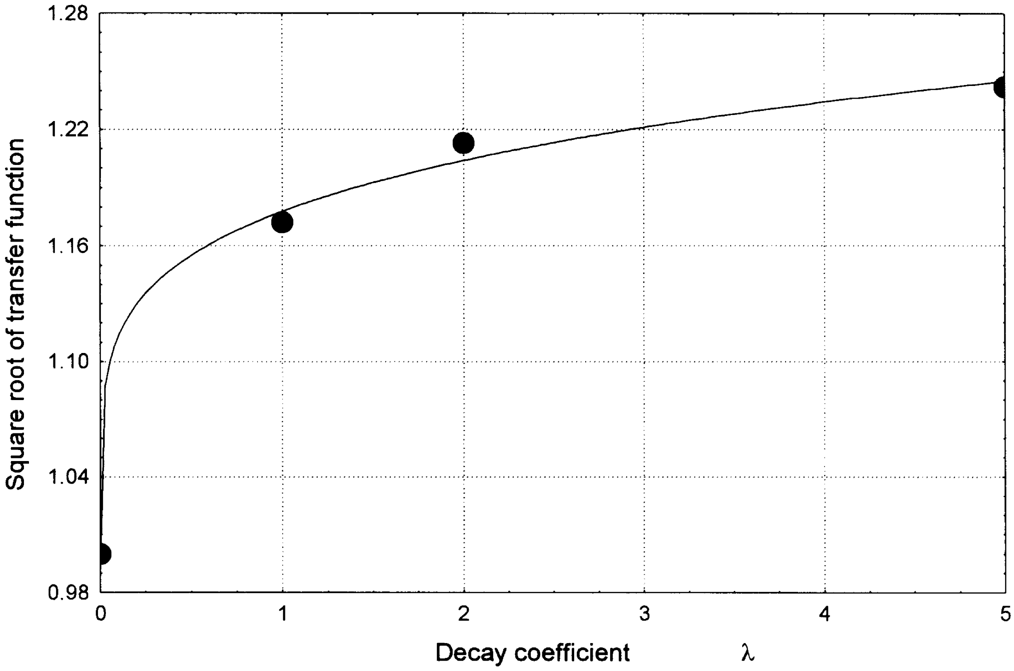

Fig. 13. Transfer function between standard deviations of normal vertical stress in 2-D and 3-D states.

loading and unit forces acting in the statistically homogeneous half-plane and half-space are considered. As the result of imposed constrains due to the plane strain assumption, the additional body and surface forces are induced. In order to determine them, additional equations in a form of constrain relations are proposed.

The analysis performed allows one to present the stress tensor for a 3-D analysis as a sum of two components. The first one reflects the plane strain state stress tensor, but given in the form of the 3-D random field. This term allows for incorporating spatial, 3-D soil variability in a 2-D analysis. The second component can be treated as a correction term and it represents the longitudinal influence of a 3-D analysis.

From the analysis performed the following conclusions can be drawn:

(1) The reduction of dimensions in the stochastic medium is possible and a spatial variability of material properties can be included in the plane strain analysis.
(2) The conditions for applying plane strain analysis in a 3-D soil medium should be considered in two aspects:
   (a) Assumptions concerning a random field of the input parameters in the longitudinal direction,
   (b) Analysis of fictitious body and surface forces induced by the plane strain assumption.
(3) The stress tensor in a 3-D soil medium can be approximated as a sum of the stress tensor in the plane strain analysis and some factor involving the influence of the third dimension.

Though the proposed constrain equations in this paper were introduced on the basis of loading due to gravity, they are also valid for the case of uniformly distributed external loading of infinite length. Only limited numerical results are presented in the paper. However, the proposed approach can be regarded as a framework for further numerical research aimed at application to a variety of geotechnical problems, for which the plane strain state is assumed.

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